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# Cyclic Proofs and Jumping Automata

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## Abstract

We consider a fragment of a cyclic sequent proof system for Kleene algebra, and we see it as a computational device for recognising languages of words. The starting proof system is linear and we show that it captures precisely the regular languages. When adding the standard contraction rule, the expressivity raises significantly: the system captures exactly the class of deterministic logspace languages. We prove this result by introducing as an intermediary model a new notion of multihead finite automata where heads can jump.

**2012 ACM Subject Classification** Theory of computation → Logic

**Keywords and phrases** Cyclic proofs, regular languages, multihead automata, transducers

**Related Version** An earlier version of this paper appears in Proc. FSTTCS 2019; we include more proofs here, and we prove that cyclic proofs actually capture all DLOGSPACE languages—something we had conjectured not to be true in the FSTTCS paper.

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## 1 Introduction

Cyclic proof systems have received much attention in the recent years. Proofs in such systems are graphs rather than trees, and they must satisfy a certain validity criterion.

Such systems have been proposed for instance by Brotherston and Simpson in the context of first order logic with inductive predicates [4], as an alternative to the standard induction schemes. The infinite descent principles associated to cyclic proofs are in general at least as powerful as the standard induction schemes, but the converse is a delicate problem. It was proven only recently that it holds in certain cases [3, 17], and that there are also cases where cyclic proofs are strictly more expressive [2].

Cyclic proof systems have also been used in the context of the  $\mu$ -calculus [8], where we have inductive predicates (least fixpoints), but also coinductive predicates (greatest fixpoints), and alternation of those. Proof theoretical aspects such as cut-elimination were studied from the linear logic point of view [11, 10], and these systems were recently used to obtain constructive proofs of completeness for Kozen’s axiomatisation [9, 1].

Building on these works, Das and Pous considered the simpler setting of Kleene algebra, and proposed a cyclic proof system for regular expression containments [6]. The key observation is that regular expressions can be seen as  $\mu$ -calculus formulas using only a single form of fixpoint: the definition of Kleene star as a least fixpoint ( $e^* = \mu x.1 + e \cdot x$ ). Their system is based on a non-commutative version of  $\mu$ MALL [10], and it is such that a sequent  $e \vdash f$  is derivable if and only if the language of  $e$  is contained in that of  $f$ . This work eventually led to an alternative proof of left-handed completeness for Kleene algebra [5].

In the latter works, it is natural to consider regular expressions as datatypes [12], and proofs of language containments as total functions between those datatypes [13]. Such a computational interpretation of cyclic proofs was exploited to prove cut-elimination in [7].

We follow the same approach here, focusing on an even simpler setting: our sequents essentially have the shape  $A^* \vdash 2$ , where  $A$  is a finite alphabet and  $2$  is a type (or formula) for Boolean values. Cyclic proofs no longer correspond to language containments: they give rise to functions from words to Booleans, *i.e.*, formal languages. We characterise the class of languages that arise from such proofs.

If we keep a purely linear proof system, as in [6, 7], we show that we obtain exactly the regular languages. In contrast, if we allow the contraction rule, we can express non-regular languages. We show that in this case, we obtain exactly the deterministic logarithmic space languages (DLOGSPACE). This is done by introducing a new class of automata, which we call *jumping multihead automata*<sup>1</sup>. Intuitively, when reading a word, a multihead automaton may only move its heads forward, letter by letter, while a jumping multihead automaton also has the possibility to let a given head jump to the position of another head. This gives the opportunity to record positions in the word, and to repeatedly analyse the suffixes starting from those positions. Cyclic proofs translate naturally into this new model that is in fact equivalent to the two-way multihead automata that were studied in the literature [14] and characterise DLOGSPACE.

### Outline.

We define our cyclic proof system and its computational interpretation in Sect. 2. Then we define jumping multihead automata and show they define the same languages as two-way multihead automata in Sect. 3. We prove the equivalence between the two models in Sect. 4 (Thm. 22), from which the characterisations of DLOGSPACE and regular languages follow. We discuss directions for future work in Sect. 5.

### Notation.

Given sets  $X, Y$ , we write  $X \times Y$  for their Cartesian product,  $X \uplus Y$  for their disjoint union, and  $X^*$  for the set of finite sequences (lists) over  $X$ . Given such a sequence  $l$ , we write  $|l|$  for its length and  $l_i$  or  $l(i)$  for its  $i^{\text{th}}$  element. We write  $\mathbb{B}$  for the set  $\{\mathbf{ff}, \mathbf{tt}\}$  of Booleans, and  $\langle x, y, z \rangle$  for tuples. We use commas to denote concatenation of both sequences and tuples, and  $\epsilon$  to denote the empty sequence. We write  $\text{Im}(f)$  for the image of a function  $f$ .

## 2 Infinite proofs and their semantics

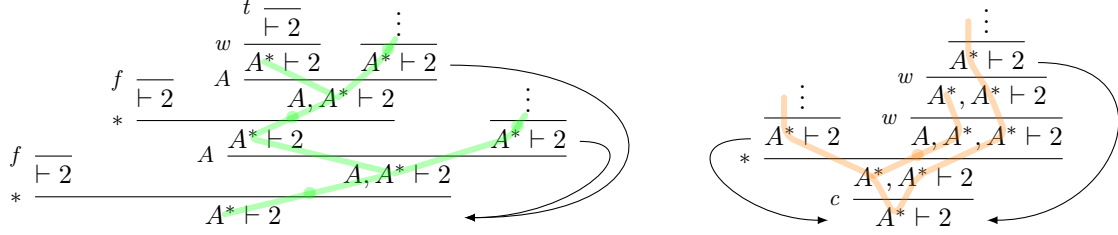
We let  $a, b$  range over the letters of a fixed and finite alphabet  $A$ . We work with only two *types* (or *formulas*): the type  $A$  of letters, and the type  $A^*$  of words. We let  $e, f$  range over types, and  $E, F$  range over finite sequences of types. Given such a sequence  $E = e_1, \dots, e_n$ , we write  $[E]$  for the set  $e_1 \times \dots \times e_n$ .

We define a sequent proof system, where sequents have the shape  $E \vdash 2$ , and where proofs of such sequents denote functions from  $[E]$  to  $\mathbb{B}$ , *i.e.* subsets of  $[E]$ .

<sup>1</sup> This new class should not be confused with the *jumping finite automata* introduced by Meduna and Zemek [15], which are not multihead.

$$\begin{array}{c}
w \frac{E, F \vdash 2}{E, e, F \vdash 2} \quad c \frac{E, e, e, F \vdash 2}{E, e, F \vdash 2} \quad A \frac{(E, F \vdash 2)_{a \in A}}{E, A, F \vdash 2} \quad * \frac{E, F \vdash 2 \quad E, A, A^*, F \vdash 2}{E, A^*, F \vdash 2} \quad t \frac{}{\vdash 2} \quad f \frac{}{\vdash 2}
\end{array}$$

■ **Figure 1** The rules of  $C$ .



■ **Figure 2** Two regular preproofs; only the one on the left is valid.

## 2.1 Infinite proofs

We now define the cyclic proof system whose six inference rules are given in Fig. 1. In addition to two *structural rules* (weakening and contraction), we have a left introduction rule for each type, and two right introduction rules for Boolean constants. Note that there is no exchange rule, which explains why the structural and left introduction rules use two sequences  $E$  and  $F$  rather than a single one.

The left introduction rule for type  $A^*$  corresponds to an unfolding rule, looking at  $A^*$  as the least fixpoint expression  $\mu X.(1 \uplus A \times X)$  (e.g., from  $\mu$ -calculus). The left premiss intuitively corresponds to the case of an empty list, while the right premiss covers the case of a non-empty list. Except from weakening and contraction, those rules form a very small fragment of those used for Kleene algebra in [7] (interpreting  $A$  as a sum  $1 + \dots + 1$  with  $|A|$  elements and 2 as the binary sum  $1 + 1$ ).

Note that we are not interested in *provability* in the present paper: every sequent can be derived trivially, using weakenings and one of the two right introduction rules. The objects of interest are the proofs themselves; this explains why we have two axioms for proving the sequent  $\vdash 2$ : they correspond to two different proofs.

We set  $B = A \uplus \{0, 1\}$ . A (possibly infinite) *tree* is a non-empty and prefix-closed subset of  $B^*$ , which we view with the root,  $\epsilon$ , at the bottom; elements of  $B^*$  are called *addresses*.

► **Definition 1.** A preproof is a labelling  $\pi$  of a tree by sequents such that, for every node  $v$  with children  $v_1, \dots, v_n$ , the expression  $\frac{\pi(v_1) \dots \pi(v_n)}{\pi(v)}$  is an instance of a rule from Fig. 1. A preproof is *regular* if it has finitely many distinct subtrees, i.e. it can be viewed as the unfolding of a finite graph. A preproof is *affine* if it does not use the  $c$ -rule.

If  $\pi$  is a preproof, we note  $\text{Addr}(\pi)$  its set of addresses, i.e. its underlying tree. The formulas appearing in lists  $E, F$  of any rule instance are called *auxiliary formulas*. The non auxiliary formula appearing in the conclusion of a rule is called the *principal formula*.

A  $*$  address in a preproof  $\pi$  is an address  $v$  which is the conclusion of a  $*$  rule in  $\pi$ .

Two examples of regular preproofs are depicted in Fig. 2. The alphabet  $A$  is assumed to have exactly two elements, so that the  $A$  rule is binary. Backpointers are used to denote

circularity: the actual preproofs are obtained by unfolding the graphs. The preproof on the right might look suspicious: it never uses the axioms  $t$  or  $f$ . In fact, only the one on the left satisfies the validity criterion which we define below. Before doing so, we need to define a notion of thread, which are the branches of the shaded trees depicted on the preproofs. Intuitively a thread follows a star formula occurrence along a branch of the proof. First we need to define parentship and ancestor relations.

► **Definition 2.** A [star] position in a preproof  $\pi$  is a pair  $\langle v, i \rangle$  consisting of an address  $v$  and an index  $i \in [0; |E| - 1]$ , where  $\pi(v) = E \vdash 2$  [and  $E_i$  is a star formula]. A position  $\langle w, j \rangle$  is the parent of a position  $\langle v, i \rangle$  if  $|v| = |w| + 1$  and, looking at the rule applied at address  $w$  the two positions point at the same place in the lists  $E, F$  of auxiliary formulas, or at the formula  $e$  when this is the contraction rule, or at the principal formula  $A^*$  when this is the  $*$  rule and  $v = w1$ . We write  $\langle v, i \rangle \triangleleft \langle w, j \rangle$  in the former cases, and  $\langle v, i \rangle \triangleleft^* \langle w, j \rangle$  in the latter case. Position  $\langle w, j \rangle$  is an ancestor of  $\langle v, i \rangle$  when those positions are related by the transitive closure of the parentship relation.

The graph of the parentship relation is depicted in Fig. 2 using shaded thick lines and an additional bullet to indicate when we pass principal star steps ( $\triangleleft^*$ ). Note that in the  $*$  rule, the principal formula occurrence  $A^*$  is not considered as a parent of the occurrence of  $A$  in the right premiss.

We can finally define threads and the validity criterion.

► **Definition 3.** A thread is a possibly infinite branch of the ancestry graph. A thread is principal when it visits a  $*$  rule through its principal formula. A thread is valid if it is principal infinitely often.

In the first preproof of Fig. 2, the infinite green thread  $\langle \epsilon, 0 \rangle \triangleright \langle 1, 1 \rangle \triangleright \langle 11, 0 \rangle \triangleright \langle 111, 1 \rangle \triangleright \langle 1111, 0 \rangle \dots$  is valid, as well as every other infinite thread. There is no valid thread in the second preproof: taking a principal step forces the thread to terminate.

► **Definition 4.** A preproof is valid if every infinite branch contains a valid thread. A proof is a valid preproof. We write  $\pi : E \vdash 2$  when  $\pi$  is a proof whose root is labelled by  $E \vdash 2$ .

In the examples from Fig. 2, only the preproof on the left is valid, thanks to the infinite green thread. The second preproof is invalid: infinite threads along the (infinitely many) infinite branches are never principal.

This validity criterion is essentially the same as for the system LKA [7], which in turn is an instance of the one used for  $\mu$ MALL [10]: we just had to extend the notion of ancestry to cover the contraction rule. Note however that the presence of this rule induces some subtleties. For instance, while in the cut-free fragment of LKA, a preproof is valid if and only if it is *fair* (i.e. every infinite branch contains infinitely many  $*$  steps [7, Prop. 8]), this is no longer true with contraction: the second preproof from Fig. 2 is fair and invalid.

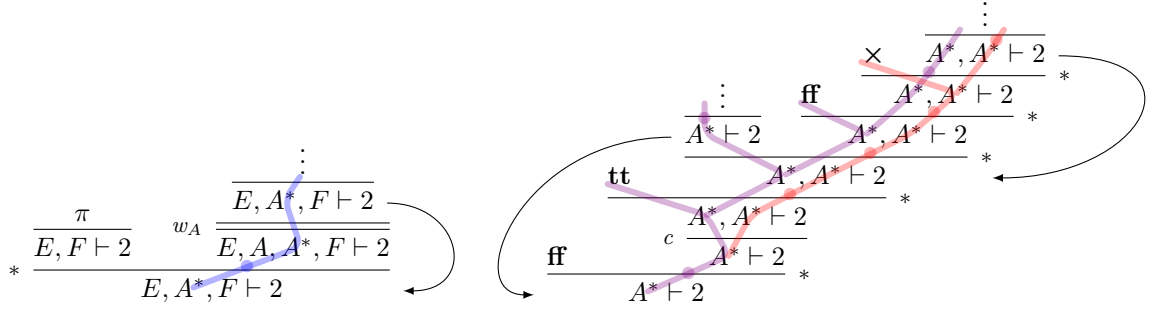
In the affine case, due to the fragment we consider here, and since we do not include cut, the situation is actually trivial:

► **Proposition 5.** Every affine preproof is valid.

## 2.2 Computational interpretation of infinite proofs

We now show how to interpret a proof  $\pi : E \vdash 2$  as a function  $[\pi] : [E] \rightarrow \mathbb{B}$ . Since proofs are not well-founded, we cannot reason directly by induction on proofs. We use instead the following relation on partial computations, which we prove to be well-founded thanks to the validity criterion.





■ **Figure 3** Weakening stars (Prop. 10).

■ **Figure 4** A regular proof for  $\{a^{2^n} \mid n \in \mathbb{N}\}$ .

deterministic automaton because of the fifth line: when reading an  $a$ , the state  $1a$  decides to accept immediately, whatever the remainder of the word. We can nevertheless deduce from those equations that  $\epsilon$  recognises the language  $A^*aaA^*$ .

Trying to perform such computations on the invalid preproof on the right in Fig. 2 gives rise to non-terminating behaviours, e.g.,  $[\epsilon](\epsilon) \rightsquigarrow [0](\epsilon, \epsilon) \rightsquigarrow [00](\epsilon) \rightsquigarrow \dots$  and  $[\epsilon](x :: q) \rightsquigarrow [0](x :: q, x :: q) \rightsquigarrow [01](x, q, x :: q) \rightsquigarrow [010](q, x :: q) \rightsquigarrow [0100](x :: q) \rightsquigarrow \dots$

Before studying a more involved example, we prove the following property:

► **Proposition 10.** *The weakening rule ( $w$ ) is derivable in a way that respects regularity, affinity, existing threads, and the semantics.*

**Proof.** When the weakened formula is  $A$ , it suffices to apply the  $A$  rule and to use the starting proof  $|A|$  times. When the weakened formula is  $A^*$ , assuming a proof  $\pi : E, F \vdash 2$ , we construct the proof in Fig. 3. The step marked with  $w_A$  is the previously derived weakening on  $A$ . The preproof is valid because this step does preserve the blue thread. ◀

As a consequence, the full proof system is equivalent to the one without weakening. We shall see that the system would remain equally expressive with the addition of an exchange rule (see Rem. 26 below), but that the contraction rule instead plays a crucial role and changes the expressive power, going from regular languages to DLOGSPACE languages.

Let us conclude this section with an example beyond regular languages: we give in Fig. 4 a proof whose semantics is the language of words over a single letter alphabet, whose length is a power of two (a language which is not even context-free). Since the alphabet has a single letter, the  $A$  rule becomes a form of weakening, and we apply it implicitly after each  $*$  step. We also abbreviate subproofs consisting of a sequence of weakenings followed by one of the two axioms by **tt**, **ff**, or just  $\times$  when it does not matter whether we return true or false.

Writing  $n$  for the word of length  $n$  and executing the proof on small numbers, we observe

$$\begin{aligned}
 [\epsilon](0) &= [0]() = \mathbf{ff} \\
 [\epsilon](1) &= [1](0) = [10](0, 0) = [100](0) = \mathbf{tt} \\
 [\epsilon](2) &= [1](1) = [10](1, 1) = [101](1, 0) = [1010](1) = [\epsilon](1) = \mathbf{tt} \\
 [\epsilon](3) &= [1](2) = [10](2, 2) = [101](2, 1) = [1011](2, 0) = \mathbf{ff} \\
 [\epsilon](4) &= [1](3) = [10](3, 3) = [101](3, 2) = [1011](3, 1) = [10111](3, 0) = [101111](2, 0) \\
 &= [101](2, 0) = [1010](2) = [\epsilon](2) = \mathbf{tt}
 \end{aligned}$$

More generally, the idea consists in checking that the given number can be divided by two repeatedly, until we get 1. To divide a number represented in unary notation by two, we copy

that number using the contraction rule, and we consume one of the copies twice as fast as the other one (through the three instances of the  $*$  rule used at addresses 101, 1011, and 10111); if we reach the end of one copy, then the number was even, the other copy precisely contains its half, and we can proceed recursively (through the backpointer on the left), otherwise the number was odd and we can reject. The subproof at address 101110 is never explored: we would be in a situation where the slowly consumed copy gets empty before the other one.

Finally note that every (even undecidable) language can be represented using an infinite (in general non regular) proof: apply the left introduction rules eagerly, and fill in the left premisses of the  $*$  rules using the appropriate axiom.

### 3 Jumping multihead automata

Now we introduce the model of Jumping Multihead Automata (JMA) and establish its equivalence with the usual two-way multihead model investigated in the literature. We will give direct translations between JMA and cyclic proofs in Sect. 4.

#### 3.1 Definition and semantics of JMAs

Let  $A$  be a finite alphabet and  $\triangleleft \notin A$  be a fresh symbol. We note  $A_{\triangleleft} = A \cup \{\triangleleft\}$ .

► **Definition 11.** A jumping multihead automaton (JMA) is a tuple  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$  where:

- $S$  is a finite set of states;
- $k \in \mathbb{N}$  is the number of heads;
- $s_0 \in S$  is the initial state;
- $s_{acc} \in S$  and  $s_{rej} \in S$  are final states, respectively accepting and rejecting;
- $\delta : S_{trans} \times (A_{\triangleleft})^k \longrightarrow S \times Act^k$  is the deterministic transition function, where  $S_{trans} \triangleq S \setminus \{s_{acc}, s_{rej}\}$  is the set of non-final states, and  $Act \triangleq \{\ddot{\cdot}, \blacktriangleright\} \cup \{J_1, J_2, \dots, J_k\}$ .

In the transition function, symbols  $\ddot{\cdot}$  and  $\blacktriangleright$  stand for “stay in place” and “move forward” respectively, and action  $J_i$  stands for “jump to the position of head number  $i$ ”. Intuitively, if the machine is in state  $s$ , each head  $j$  reads letter  $\vec{a}(j)$ , and  $\delta(s, \vec{a}) = (s', \alpha)$ , then the machine goes to state  $s'$  and each head  $j$  performs the action  $\alpha(j)$ . Accordingly, to guarantee that the automaton does not try to go beyond the end marker of the word, we require that if  $\delta(s, \vec{a}) = (s', \alpha)$ , then for all  $j \in \llbracket 1, k \rrbracket$  with  $\vec{a}(j) = \triangleleft$  we have  $\alpha(j) \neq \blacktriangleright$ .

A *configuration* of a JMA  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$  is a triple  $c = (w, s, p)$  where  $w$  is the input word,  $s \in S$  is the current state, and  $p = (p_1, \dots, p_k) \in \llbracket 0, |w| \rrbracket^k$  gives the current head positions. If the position  $p_i$  is  $|w|$  then the head  $i$  is scanning the symbol  $\triangleleft$ .

The initial configuration on an input word  $w$  is  $(w, s_0, (0, \dots, 0))$ . Let  $w = a_0 a_1 \dots a_{n-1}$  be the input and  $a_n = \triangleleft$ . Let  $(w, s, (p_1, \dots, p_k))$  be a configuration with  $s \in S_{trans}$ , and  $(s', (x_1, \dots, x_k)) = \delta(s, (a_{p_0}, \dots, a_{p_k}))$  be given by the transition function. Then the successor configuration is defined by  $(w, s', (p'_1, \dots, p'_k))$ , where for all  $i \in \llbracket 1, k \rrbracket$   $p'_i$  depends on  $x_i$  in the following way:

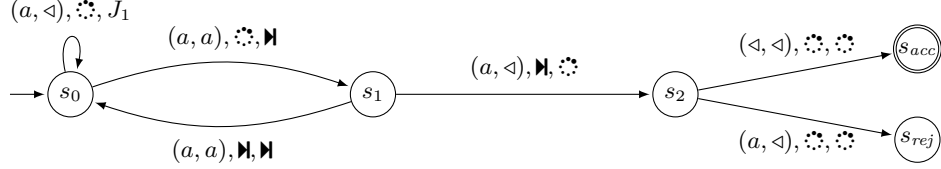
$$(1) \ p'_i = p_i \text{ if } x_i = \ddot{\cdot} \qquad (2) \ p'_i = p_i + 1 \text{ if } x_i = \blacktriangleright \qquad (3) \ p'_i = p_j \text{ if } x_i = J_j$$

A configuration  $(w, s, p)$  is *final* if  $s \in \{s_{acc}, s_{rej}\}$ . It is accepting (resp. rejecting) if  $s = s_{acc}$  (resp.  $s = s_{rej}$ ). A *run* of a JMA  $\mathcal{M}$  on  $w$  is a sequence of configurations  $c_0, c_1, \dots, c_r$  on  $w$  where  $c_0$  is the initial configuration, and  $c_{i+1}$  is the successor configuration of  $c_i$  for all  $i$ . If  $c_r$  is rejecting (resp. accepting), we say that the word  $w$  is rejected (resp. accepted) by



$\mathcal{M}$ . We say that  $\mathcal{M}$  *terminates* on  $w$  if there is a maximal finite run of  $\mathcal{M}$  on  $w$ , ending in a final configuration. The *language* of  $\mathcal{M}$ , noted  $L(\mathcal{M})$ , is the set of finite words accepted by  $\mathcal{M}$ , *i.e.* the set of words  $w \in A^*$  such that  $\mathcal{M}$  has an accepting run on  $w$ .

► **Example 12.** The language  $L = \{a^{2^n} \mid n \in \mathbb{N}\}$  can be recognised by the following JMA with two heads. (Missing transitions all go to the rejecting final state.)



The idea behind the automaton is similar as the proof given in Fig. 4: one head advances at twice the speed of the other. When the fast head reaches the end of the word, it either rejects if the length is odd and at least 2, or jumps to the position of the slow head located in the middle of the word. From there, the automaton proceeds recursively.

Notice that on an input word  $u$ , three scenarios are possible: the automaton accepts by reaching  $s_{acc}$ , rejects by reaching  $s_{rej}$ , or rejects by looping forever. In order to translate JMAs into cyclic proofs, whose validity criterion ensures termination, it is convenient to forbid the last scenario. We ensure such a property by a syntactic restriction on the transition structure of JMAs.

The *transition graph* of a JMA  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$  is the labelled graph  $G_{\mathcal{M}} = (S, E)$ , where the vertices are states  $S$ , and the set of edges is  $E \subseteq S \times S \times Act^k$ , defined by  $E = \{(s, s', \alpha) \mid \exists \vec{a} \in (A_d)^k, \delta(s, \vec{a}) = (s', \alpha)\}$ .

A JMA  $\mathcal{M}$  is *progressing* if for every cycle  $e_1 e_2 \dots e_l$  in its transition graph, where  $e_i = (s_i, s_{i+1}, \alpha_i)$  for each  $i \in \llbracket 1, l \rrbracket$  and  $s_{l+1} = s_1$ , there exists a head  $j \in \llbracket 1, k \rrbracket$  with  $\alpha_1(j) \alpha_2(j) \dots \alpha_l(j) \in (\cdot^* \cdot \mathbf{H} \cdot \cdot^*)^+$ . (Intuitively we require that for every loop, one of the head does not jump during this loop and moves forward at least once).

The JMA from Ex. 12 always terminates, but it is not progressing due to the loop on the initial state. It could easily be modified into a progressing JMA by introducing a new intermediary state instead of looping on  $s_0$ . In fact, even in cases where a JMA can indefinitely loop on some inputs, one can always turn it into a progressing one recognising the same language. Hence all JMAs are assumed to be progressing from now on.

► **Lemma 13.** *Every JMA can be converted into a progressing JMA with the same language.*

**Proof.** We use the fact that the number of possible configurations on a given word  $w$  is bounded polynomially in the length of  $w$ . We add heads to the JMA that just advance counting up until this bound, making the JMA progressing. Details are given in the Appendix. ◀

► **Lemma 14.** *Given a JMA  $\mathcal{M}$ , we can check in NL whether  $\mathcal{M}$  is progressing. If  $\mathcal{M}$  is progressing, then it terminates on all words.*

### 3.2 Expressive power of JMAs

Write  $JMA(k)$  for the set of languages expressible by a progressing JMA with  $k$  heads. JMAs encode precisely the DLOGSPACE languages; one-head JMAs capture exactly the regular languages.

► **Theorem 15.**  $\bigcup_{k \geq 1} JMA(k) = \text{DLOGSPACE}$

► **Lemma 16.**  $JMA(1) = \text{REG}$ .

The lemma is trivial, and the forward direction of Thm. 15 is relatively easy:

► **Lemma 17.**  $\bigcup_{k \geq 1} JMA(k) \subseteq \text{DLOGSPACE}$ .

**Proof.** It is straightforward to translate a JMA with  $k$  heads into a Turing machine using space  $O(\log^k(n))$ , by remembering the position of the heads. ◀

To obtain the other direction of Thm. 15, we go through a notion of (non-jumping) multihead automata that has already been investigated in the literature [14]. They consist of automata with a fixed number of heads ( $k$ ) that can either only go from left to right, (one-way automata,  $1DFA(k)$ ), or in both directions (two-way automata,  $2DFA(k)$ ). We briefly compare JMAs to those automata, starting with the one-way case.

First of all, it is clear that for all  $k \geq 1$ ,  $1DFA(k) \subseteq JMA(k)$  (in particular, because  $1DFA$ s can be assumed to be progressing without increasing the number of heads).

► **Remark 18.** Since emptiness, universality, regularity, inclusion and equivalence are undecidable for  $1DFA$ s with 2 heads [14], these problems are also undecidable for JMAs with 2 heads.

Concerning two-way automata ( $2DFA$ ) it is known that  $\bigcup_{k \geq 1} 2DFA(k) = \text{DLOGSPACE}$  [14], so that by Lem. 17 every JMA can be translated into a deterministic multihead two-way automaton, not necessarily preserving the number of heads. We prove the converse direction with a direct transformation of a two-way multihead automaton into a jumping multihead one. We start by giving an example of a seemingly 2-way behaviour that can be simulated by JMAs.

► **Example 19.** The palindrom language  $\mathcal{L} = \{w \mid w \in A^* \wedge w = w^R\}$  where  $w^R$  denotes the reverse of  $w$  is recognizable by a JMA with 4 heads. Let us call these 4 heads  $h_0$ ,  $h_{lin}$ ,  $h_{lin}^-$  and  $h_{temp}$  in the following. The head  $h_0$ , will always stay at the beginning of the word to allow other heads to jump back to this position. The head  $h_{lin}$  will linearly read the word from left to right. The head  $h_{lin}^-$  will simulate the linear reading of the word from right to left by doing multiple jumps. The head  $h_{temp}$  will help  $h_{lin}^-$  to find the position of its next move.

In the initial configuration of the automaton, all the reading heads are locating on the first letter of the input word. Then, each time  $h_{lin}$  is reading the  $i$ -th letter we do as follow:

- $h_{lin}^-$  and  $h_{temp}$  jump respectively on  $h_0$  and  $h_{lin}$  and then  $h_{temp}$  moves one step right. They are exactly  $i$  letters apart.
- $h_{lin}^-$  and  $h_{temp}$  move right synchronously until  $h_{temp}$  is reading the end symbol of the word  $\triangleleft$  (i.e. the  $(n+1)$ -th letter). Then  $h_{lin}^-$  is reading the  $(n+1-i)$ -th letter.

The automaton compares the letters read by  $h_{lin}$  and  $h_{lin}^-$ . If they are different it halts and rejects the word. Otherwise  $h_{lin}$  moves right. If it reaches the end symbol of the word  $\triangleleft$  then the automaton halts and accepts, otherwise we do the same process again.

The automaton recognizing the palindrom language works by simulating a head that reads the word from the right to the left thanks to an auxiliary head and multiple jumps. We can generalize this process to simulate any two-way multihead automaton.

► **Lemma 20.** Let  $\mathcal{M}$  be a two-way (deterministic) multihead automaton with  $k$  heads. Then there exists  $\mathcal{M}'$  a JMA with  $2k+2$  heads such that  $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{M}')$ .

**Proof.** We construct  $\mathcal{M}'$  as follows.

For each head  $h$  of  $\mathcal{M}$  we add another reading head  $h'$  that will be always kept at the symmetric position of  $h$  (*i.e.* when  $h$  will be reading the  $i$ -th letter,  $h'$  would be reading the  $n - i + 1$ -th letter). Moreover we add a head  $h_0$  that will always stay at the beginning of the word and a head  $h_{temp}$  that we will use to make a given head move one letter left.

Each time a head  $h$  moves (left or right) in  $\mathcal{M}$ , in  $\mathcal{M}'$  the corresponding head  $h$  will do the same move and its symmetric head  $h'$  the inverse move. We always begin by doing the right move and then use the fact that we have a head at the symmetric position to find the position where the second head should move with  $h_{temp}$ . For simplicity let's assume that  $h$  moves right so that  $h'$  should move left. If  $h$  is reading the  $i$ -th letter then we have to find the  $n - i + 1$ -th letter. We proceed as follow:

- $h'$  and  $h_{temp}$  jump respectively on  $h_0$  and  $h$  and then  $h_{temp}$  moves one step right. They are exactly  $i$  letters apart.
- $h'$  and  $h_{temp}$  move right synchronously until  $h_{temp}$  is reading the end symbol of the word  $\triangleleft$  (*i.e.* the  $n + 1$ -th letter). Then  $h'$  is reading the  $n + 1 - i$ -th letter.

◀

Since it was shown in [18] that  $2DFA$ s are strictly more expressive than  $1DFA$ s, it is also the case that  $JMA$ s are strictly more expressive than  $1DFA$ s. We refine this result here, and show that  $JMA$ s with 2 heads can already compute languages that are not computable by  $1DFA$ s with any number of heads.

► **Proposition 21.** *For all  $k \geq 1$ ,  $JMA(2) \not\subseteq 1DFA(k)$ .*

**Proof.** It is proven in [18] that  $(1DFA(k))_{k \in \mathbb{N}}$  forms a strict hierarchy, by defining a language  $L_b$  that is recognisable by a  $1DFA$  with  $k$  heads if and only if  $b < \binom{k}{2}$ . We slightly modify these languages so that they become expressible with a two-head  $JMA$  while keeping the previous characterisation for  $1DFA$ . Details are given in the Appendix. ◀

## 4 Equivalence between $JMA$ s and cyclic proofs

We now turn to proving the following equivalence.

► **Theorem 22.** *The languages recognised by  $JMA$ s are those recognised by regular proofs.*

Together with Thm. 15 we deduce that regular proofs recognise exactly the  $DLOGSPACE$  languages. We prove the theorem in the next two subsections, by providing effective translations between the two models. Notice that by Rem. 18, the theorem implies that for regular proofs  $\pi$ , emptiness and other basic properties of  $[\pi]$  are undecidable.

### 4.1 From $JMA$ s to cyclic proofs

Let  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$  be a jumping multihead automaton. We want to build a regular proof  $\pi_{\mathcal{M}}$  of  $A^* \vdash 2$  such that  $[\pi_{\mathcal{M}}] = L(\mathcal{M})$ . A difficulty is that heads in the automaton may stay in place, thus reading the same letter during several steps. In contrast the letters are read only once by cyclic proofs, so that we have to remember this information. We do so by labelling the sequents of the produced proof  $\pi_{\mathcal{M}}$  with extra information describing the current state of the automaton. If  $k' \in \mathbb{N}$ , let  $\mathcal{F}_{k'}$  be the set of injective functions  $[1, k'] \rightarrow [1, k]$ . A *labelled sequent* is a sequent of the form  $(A^*)^{k'} \vdash 2$  together with an extra label in  $S \times \mathcal{F}_{k'} \times (A \cup \{\square, \triangleleft\})^k$ .

The intuitive meaning of a label  $(s, f, \vec{y})$  is the following:  $s$  is the current state of the automaton,  $f$  maps each formula  $A^*$  of the sequent to a head of the automaton, and  $\vec{y}$  stores the letter that is currently processed by each head. Symbol  $\square$  is used if this letter is unknown, and the head is scheduled to process this letter and move to the right. The values intuitively provided to each  $A^*$  formula of the sequent are the suffixes to the right of the corresponding heads of the automaton. On the examples, labels will be written in grey below the sequents.

It will always be the case that if the label of  $(A^*)^{k'}$  is  $(s, f, \vec{y})$ , then  $\text{Im}(f) \subseteq \{i \mid y_i \neq \square\}$ , *i.e.* all heads reading symbols from  $A \cup \{\square\}$  correspond to a formula  $A^*$  in the sequent. We say that a sequent is *fully labelled* if its label does not contain  $\square$ .

The construction of  $\pi_{\mathcal{M}}$  will proceed by building gadgets in the form of proof trees, each one (apart from the initial gadget) connecting a labelled sequent in the conclusion to a finite set of labelled sequents in the hypotheses. If some labelled sequents in the hypotheses have already been encountered, we simply put back pointers to their previous occurrence. Since the number of labelled sequents is finite, this process eventually terminates and yields a description of  $\pi_{\mathcal{M}}$ .

When describing those gadgets we abbreviate sequences of inference steps or standalone proofs using double bars labelled with the involved rule names.

**Initial gadget.** The role of the initial gadget is to reach the first labelled sequent from the conclusion  $A^* \vdash 2$ . It simply creates  $k$  identical copies of  $A^*$ . This expresses the fact that the initial configuration is  $\langle w, s_0, (0, 0, \dots, 0) \rangle$ . We note  $\text{id}_k$  the identity function on  $\llbracket 1, k \rrbracket$ . The initial labelled sequent is  $(A^*)^k \vdash 2$  together with label  $(s_0, \text{id}_k, (\square, \dots, \square))$ .

The initial gadget is as follows:

$$c, \dots, c \frac{(A^*)^k \vdash 2 \quad s_0, \text{id}_k, (\square, \dots, \square)}{A^* \vdash 2}$$

**Reading gadget.** Every time the label  $(s, f, \vec{y})$  of the current address is not fully labelled, we use the gadget  $\text{read}_i$ , where  $i = \min\{j \mid \vec{y}(j) = \square\}$  to process the first unknown letter.

We note  $i' = f^{-1}(i)$  the position of the  $A^*$  formula corresponding to head  $i$  and define the gadget  $\text{read}_i$  as follows:

$$\frac{(A^*)^{k'-1} \vdash 2 \quad A \frac{\left( \frac{(A^*)^{k'} \vdash 2}{s, f, (y_1, \dots, y_{i-1}, a, \dots, y_k)} \right)_{a \in A}}{(A^*)^{i'-1}, A, A^*, (A^*)^{k'-i'} \vdash 2}}{(A^*)^{k'} \vdash 2} \quad \text{where } f'(x) = \begin{cases} f(x) & \text{if } 1 \leq x < i' \\ f(x+1) & \text{if } i' \leq x \leq k' - 1 \end{cases} \quad *$$

$$\frac{(A^*)^{k'} \vdash 2 \quad s, f, (y_1, \dots, y_{i-1}, \square, \dots, y_k)}{(A^*)^{k'} \vdash 2}$$

**Transition gadget.** Thanks to the  $\text{read}_i$  gadgets, we can now assume we reach a fully labelled sequent, with label of the form  $(s, f, (y_1, \dots, y_k))$ . If  $s \notin \{s_{acc}, s_{rej}\}$ , we use a *transition gadget*, whose general shape is as on the right below, with  $(s', \alpha) = \delta(s, (y_1, \dots, y_k))$ :

This gadget is designed such that for all  $i \in \llbracket 1, k \rrbracket$ :

- if  $\alpha(i) = \clubsuit$  then  $z_i = y_i$
- if  $\alpha(i) = \spadesuit$  then  $z_i = \square$ ,
- if  $\alpha(i) = J_j$  then  $z_i = y_j$ .

$$\delta \frac{(A^*)^{k''} \vdash 2 \quad s', f', (z_1, \dots, z_k)}{(A^*)^{k'} \vdash 2 \quad s, f, (y_1, \dots, y_k)}$$

In the last case, a contraction is used to duplicate the  $A^*$  formula corresponding to head  $j$ , and the function  $f'$  maps this new formula to head  $i$ . The occurrence of  $A^*$  corresponding to  $y_i$  is weakened (possibly after having been duplicated if another head jumped to  $i$ ).

We describe this gadget on two examples below. An element  $f : \llbracket 1, k' \rrbracket \rightarrow \llbracket 1, k \rrbracket$  is simply represented by  $f(1)f(2) \dots f(k')$ .

$$\delta(s, (a, b, \triangleleft)) = (s', (\mathbf{M}, \diamond, J_1))$$

$$\begin{array}{c} A^*, A^*, A^* \vdash 2 \\ s', 132, (\square, b, a) \\ \hline c \quad A^*, A^* \vdash 2 \\ s, 12, (a, b, \triangleleft) \end{array}$$

$$\delta(s, (c, d, e)) = (s', (J_3, \mathbf{M}, J_2))$$

$$\begin{array}{c} A^*, A^*, A^* \vdash 2 \\ s', 231, (e, \square, d) \\ \hline w, w \quad A^*, A^*, A^*, A^* \vdash 2 \\ c, c \quad \hline A^*, A^*, A^* \vdash 2 \\ s, 123, (c, d, e) \end{array}$$

Notice that it is also possible to avoid unnecessary contractions, in order to bound the number of  $A^*$  formulas in a sequent by  $k$ . The symbol  $\square$  means that the formula  $A^*$  is scheduled for a  $*$  rule, and will be immediately processed thanks to the gadget  $read_i$  as described above.

**Final gadget.** It remains to describe what happens if the current sequent is fully labelled with  $s \in \{s_{acc}, s_{rej}\}$ . In this case, we simply conclude with a **(tt)** axiom if  $s = s_{acc}$  or with a **(ff)** axiom if  $s = s_{rej}$ .

This achieves the description of the preproof  $\pi_{\mathcal{M}}$ . The following lemma expresses its correctness; we prove it in the Appendix.

► **Lemma 23.** *If  $\mathcal{M}$  is a progressing JMA, the preproof  $\pi_{\mathcal{M}}$  is valid, and  $[\pi_{\mathcal{M}}] = L(\mathcal{M})$ .*

## 4.2 From cyclic proofs to JMAs

Let  $\pi$  be a regular proof with conclusion  $A^* \vdash 2$ . Let  $k$  be the maximal number of star formulas in the sequents of  $\pi$ . We build a JMA  $\mathcal{M}$  with  $k$  heads such that  $L(\mathcal{M}) = [\pi]$ .

The idea of the construction is to store all necessary information on the current state of the computation in  $\pi$  into the state space of  $\mathcal{M}$ , besides the content of star formulas. This includes the current address in  $\pi$ , and the actual letters corresponding to the alphabet formulas, together with some information linking star formulas to heads of the automaton.

This allows  $\mathcal{M}$  to mimic the computation of  $[\pi]$  on an input  $u$ , in a similar way as the converse translation from Sect. 4.1. In particular, we keep the invariant that the value associated to each star formula is the suffix of  $u$  to the right of the corresponding head of  $\mathcal{M}$ .

**State space of  $\mathcal{M}$ .** Let  $m$  be the maximal number of alphabet formulas in the sequents of  $\pi$ . We use a register with  $m$  slots, each one possibly storing a letter from  $A$ . Let  $R = \bigcup_{i=0}^m A^i$  be the set of possible register values. An element  $b_1 \dots b_i$  of  $R$  describes the content of the  $i$  alphabet formulas of the current sequent. We denote the empty register by  $\diamond$ . Intuitively, the register needs to store the values that have been processed by the automaton, but are still unknown in the proof  $\pi$  as they are represented by alphabet formulas.

Let  $\mathcal{F}$  be the set  $\bigcup_{i=0}^k \llbracket 1, k \rrbracket^i$ . An element  $f \in \mathcal{F}$  associates to each  $A^*$  formula of a sequent the index of a head of  $\mathcal{M}$ . This allows us to keep track of the correspondence between heads of  $\mathcal{M}$  and suffixes of the input word being processed by  $\pi$ .

We define the state space of  $\mathcal{M}$  as  $S = (Addr(\pi) \times R \times \mathcal{F}) \cup \{s_{acc}, s_{rej}\}$ .

Notice that  $Addr(\pi)$  is infinite, so  $\mathcal{M}$  is an infinite-state JMA. However, if  $\pi$  has finitely many subtrees, we will be able to quotient  $Addr(\pi)$  by  $v \sim w$  if  $v$  and  $w$  correspond to the same subtree, and obtain a finite-state JMA.

If  $(v, r, f)$  is a state of  $\mathcal{M}$ , we will always have  $|r| = m'$  and  $|f| = k'$ , where  $m'$  (resp.  $k'$ ) is the number of alphabet (resp. star) formulas in  $\pi(v)$ . Moreover, for all  $i \in \llbracket 1, m' \rrbracket$ , the  $i^{th}$  alphabet formula contains the letter  $r(i)$  stored in the  $i^{th}$  slot of the register  $r$ .

The initial state is  $s_0 = (\epsilon, \diamond, 1)$ . It points to the root of  $\pi$ , with empty register, and maps the only star formula to head 1.

**Transition function of  $\mathcal{M}$ .** If  $s = (v, r, f)$  is a state of  $\mathcal{M}$ , and  $\vec{a} = (a_1, \dots, a_k)$  is the tuple of letters read by each head with  $a_i \in A_\Delta$ , we want to define  $\delta(s, \vec{a}) = (s', \alpha) \in S \times Act^k$ .

We write  $\alpha_{id}$  for the action tuple  $(\text{⋈}, \dots, \text{⋈})$  leaving each head at the same position. We write  $move_i$  (resp.  $jump_{i,j}$ ) for the element of  $Act^k$  which associates to heads  $i' \neq i$  the action  $\text{⋈}$  and to head  $i$  the action  $\text{⋈}$  (resp. jump to head  $j$ ).

First of all, if the rule applied to  $v$  in  $\pi$  is an axiom (**tt**) (resp. (**ff**)), we set  $s' = s_{acc}$  (resp.  $s_{rej}$ ) and  $\alpha = \alpha_{id}$ . This allows  $\mathcal{M}$  to stop the computation and return the same value as  $[\pi]$ . Otherwise, we define  $s' = (v', r', f')$  and  $\alpha$  depending on the rule applied to  $v$  in  $\pi$ . By Prop. 10, we can assume that the proof  $\pi$  does not use the weakening rule. Let  $m'$  (resp.  $k'$ ) be the number of alphabet (resp. star) formulas in  $\pi(v)$ .

#### Contraction rule:

We set  $v' = v0$ , and do a case analysis on the principal formula:

- $i^{th}$  alphabet formula: we set  $f' = f$ ,  $r' = r(1) \cdots r(i-1) \cdot r(i) \cdot r(i) \cdot r(i+1) \cdots r(m')$  and  $\alpha = \alpha_{id}$ .
- $i^{th}$  star formula: let  $j \in \llbracket 1, k \rrbracket$  be the smallest integer not appearing in  $f$ , corresponding to the index of the first available head. We want to allocate it to this new copy, by making it jump to the position of the head  $f(i)$ . We take  $r' = r$ ,  $f' = f(1) \cdots f(i) \cdot j \cdot f(i+1) \cdots f(k')$ , and  $\alpha = jump_{j, f(i)}$ .

#### Star rule:

Let  $i$  be the index of the principal star formula. We now want the head  $j \triangleq f(i)$  pointing on this formula to move right. The letter processed by this head will be added to the register.

- if  $\vec{a}(j) = \triangleleft$ , the head reached the end of the input. This corresponds to the left premiss of the  $*$  rule. We set  $v' = v0$ ,  $f' = f(1) \cdots f(i-1)f(i+1) \cdots f(k')$ ,  $r' = r$  and  $\alpha = \alpha_{id}$ .
- if  $\vec{a}(j) \in A$ , we set  $v' = v1$ ,  $f' = f$ ,  $\alpha = move_i$ , and  $r' = r(1) \cdots r(i')\vec{a}(i)r(i'+1) \cdots r(m')$ , where  $i'$  is the number of  $A$  formulas before the principal star formula.

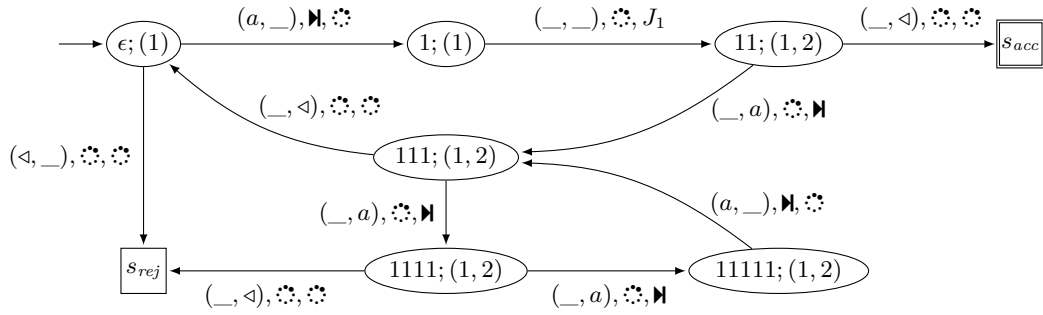
#### Alphabet rule:

Let  $i$  be the index of the principal  $A$  formula, and  $a = r(i)$  be the letter associated to it. We define  $v' = va$ ,  $f' = f$ ,  $\alpha = \alpha_{id}$ , and  $r' = r(1) \cdots r(i-1)r(i+1) \cdots r(m')$ , *i.e.* we erase the  $i^{th}$  slot.

This completes the description of the JMA  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$ .

► **Lemma 24.** *The JMA  $\mathcal{M}$  is progressing, and  $L(\mathcal{M}) = [\pi]$ .*

► **Example 25.** We can obtain a progressing JMA for the language  $L = \{a^{2^n} \mid n \in \mathbb{N}\}$  by translating the proof from Fig. 4 using the above procedure. As there are at most two star formulas in the sequents of the proof, the produced JMA has two heads. As there is only one letter in the alphabet, we can just forget the register. Similarly we consider that any **ff** part (resp. **tt** part) of the proof corresponds to the state  $s_{rej}$  (resp.  $s_{acc}$ ). Using  $\_$  for reading any symbol (a letter  $a$  or  $\triangleleft$ ), we can represent the obtained automaton as follows:



► **Remark 26.** Our encoding from regular proofs to JMAs would still work if we had included an exchange rule in the system, and the encoding from JMAs to regular proofs does not require the exchange rule. Therefore, such a rule would not increase the expressive power.

### 4.3 The affine case: regular languages

Looking at the encodings in the two previous sections, we can observe that:

- the encoding of an affine regular proof is a JMA with a single head: in absence of contraction, all sequents in proof ending with  $A^* \vdash 2$  have at most one star formula;
- the encoding of a JMA with a single head does not require contraction: this rule is used only for the initial gadget and when the action of a head is to jump on another one.

As a consequence, we have a correspondence between affine regular proofs and JMAs with a single head, whence, by Lemma 16:

► **Theorem 27.** *The regular languages are those recognisable by affine regular proofs.*

## 5 Conclusion

We have defined a cyclic proof system where proofs denote formal languages, as well as a new automata model: jumping multihead automata. We have shown that regular proofs correspond precisely to the languages recognisable by jumping multihead automata, which turn out to be the DLOGSPACE languages. Moreover we have shown that the restriction to affine regular proofs corresponds to the regular languages. We see two directions for future work.

First, we restricted to sequents of the shape  $E \vdash 2$  in order to focus on languages. The proof system we started from (LKA [7]) however makes it possible to deal with sequents of the shape  $E \vdash e$ : it suffices to include right introduction rules for the alphabet ( $A$ ) and star formulas ( $A^*$ ). By doing so, we obtain a system where proofs of  $A^* \vdash A^*$  denote *transductions*: functions from words to words. We conjecture that in the affine case, we obtain exactly the *subsequential transductions* [16], *i.e.* transductions definable by deterministic 1-way transducers. In the general case (with contraction), we would need a notion of jumping multihead transducers.

Second, we used a *cut-free* proof system. While adding the cut rule for the presented system (restricted to sequents  $E \vdash 2$ ) seems peculiar since the input and output are not of the same shape, it becomes reasonable when moving to general sequents for transductions. We have observed that we can go beyond MSO-definable transductions when doing so, even in the affine case. We would like to investigate and hopefully characterise the corresponding class of transductions.



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## A

 Appendix

► **Proposition 28** (Proposition 5 in the main text). *Every affine preproof is valid.*

**Proof.** Except for the contraction rule and the right premiss of the  $*$  rule, the length of a sequent strictly decreases when moving from the conclusion of a rule to one of its premisses. Therefore, every infinite branch of an affine preproof must pass through the right of a  $*$  rule infinitely often. By the subformula property, the principal (star) formulas of these steps must be the descendants of star formulas in the conclusion of the preproof. Since they are finitely many, at least one of the star formulas from the conclusion gives rise to a valid thread. ◀

► **Lemma 29** (Lemma 13 in the main text). *Every JMA can be converted into a progressing JMA with the same language.*

**Proof.** Let  $\mathcal{M} = \langle S, k, s_0, s_{acc}, s_{rej}, \delta \rangle$  be a JMA. We want to construct a progressing JMA  $\mathcal{M}'$  such that  $L(\mathcal{M}) = L(\mathcal{M}')$ .

For all  $w$  such that  $\mathcal{M}$  halts on  $w$ , the run  $c_0, c_1, \dots, c_r$  of  $\mathcal{M}$  on  $w$  is such that  $r < |S||w|^k$ . Indeed, for a given word  $w$ , there are only  $|S||w|^k$  distinct configurations. So if there is an accepting run  $c_0, c_1, \dots, c_r$  on  $w$  of length greater than  $|S||w|^k$  then necessarily there exists  $i \neq j$  such that  $c_i = c_j$ , meaning the automaton  $\mathcal{M}$  has entered a loop. Since  $\mathcal{M}$  is deterministic, it will stay in this loop forever, which contradicts the fact that  $\mathcal{M}$  halts on  $w$ .

We construct  $\mathcal{M}'$  by adding  $k+1$  heads to  $\mathcal{M}$ . The  $(k+1)^{th}$  head stays at the beginning of the word to allow the other heads to jump back to it. The first head reads the word letter by letter; when it reaches the final marker  $\triangleleft$ , it jumps back to the beginning and starts again. Then for all  $i$ , the  $(i+1)^{th}$  head advances each time the  $i^{th}$  reads the end symbol  $\triangleleft$  and jumps back to the beginning every time it reaches the end. Note that the  $i^{th}$  head takes exactly  $|w|^i$  steps to read the whole word. The state space  $S'$  of  $\mathcal{M}'$  is defined as  $S \times \llbracket 1, |S| \rrbracket$ , with initial state  $s'_0 = (s_0, 1)$ . The second component of  $S'$  will be called the *counter*. Each time the  $k^{th}$  head reads the end symbol, we increment the counter in addition to jumping back to the beginning. If the counter reaches  $|S|$  and needs to be incremented, the automaton  $\mathcal{M}'$  enters state  $s_{rej}$  and rejects the input.

$\mathcal{M}'$  is progressing. Indeed if there is a loop  $e_1 e_2 \dots e_l$  with  $e_i = (s_i, s_{i+1}, \alpha_i)$  in  $G_{\mathcal{M}'}$  then it corresponds to a loop of  $\mathcal{M}$  with a fixed counter value, so the  $k^{th}$  head never jumps back. Let  $i = \max\{j \in \llbracket 1, k \rrbracket \mid \exists t \alpha_t(j) = \mathbf{N}\}$ . Then the  $i^{th}$  head never jumps back. In fact if  $i < k$  and the  $i^{th}$  head had jumped back then the  $(i+1)^{th}$  head would have advanced, which contradicts the maximality of  $i$ .

$\mathcal{M}'$  recognises exactly  $L(\mathcal{M})$ . Indeed if  $w \in L(\mathcal{M})$  then there exists an accepting run of  $\mathcal{M}$  of length less than  $|S||w|^k$ , this run also exists in  $\mathcal{M}'$  and so  $w \in L(\mathcal{M}')$ . If  $w \notin L(\mathcal{M})$  then either  $\mathcal{M}$  rejects it in less than  $|S||w|^k$  steps, in which case  $\mathcal{M}'$  also rejects it, or  $\mathcal{M}'$  will reject the word after  $|S||w|^k + 1$  steps. ◀

► **Lemma 30** (Lemma 14 in the main text). *Given a JMA  $\mathcal{M}$ , we can check in NL whether  $\mathcal{M}$  is progressing. If  $\mathcal{M}$  is progressing, then it terminates on all words.*

**Proof.** To witness that an input JMA  $\mathcal{M}$  is not progressing, it suffices to guess on-the-fly a loop in the transition graph violating the condition defining progressing automata. Notice that the memory needed to verify that the loop violates the condition is  $\Theta(k)$ , since for each head, one needs to remember whether it has already violated the condition, and if not whether it has already moved to the right. The transition table of  $\mathcal{M}$  is of size exponential in  $k$ , so this memory of  $\Theta(k)$  is indeed logarithmic in the input size. Since  $\text{NL} = \text{coNL}$ , this yields a NL algorithm to verify that a JMA is progressing.

We now show that if  $\mathcal{M}$  is a progressing JMA, it terminates on all words. Assume by contradiction that there is a word  $w$  of length  $n$  such that  $\mathcal{M}$  does not terminate on  $w$ . Since its transition function is total, it means that  $\mathcal{M}$  has an infinite run  $\rho$  on  $w$ . Let  $I \subseteq \llbracket 1, k \rrbracket$  be the set of heads that advance infinitely many times in  $\rho$ . We can choose a factor  $\tau$  of  $\rho$  such that each head from  $I$  advances at least  $n + 2$  times in  $\tau$ , heads not in  $I$  do not advance in  $\tau$ , and additionally the first and last state of  $\tau$  are identical. Since  $\tau$  corresponds to a loop in the transition graph of  $\mathcal{M}$ , and  $\mathcal{M}$  is progressing, there must be a head  $j \in I$  that does not jump during  $\tau$ . This means this head  $j$  advances  $n + 2$  times without jumping, which is impossible on the word  $w$  of length  $n$ . We reached a contradiction, thereby proving that a progressing JMA must terminate on all words.  $\blacktriangleleft$

► **Lemma 31** (Lemma 16 in the main text).  $JMA(1) = \text{REG}$ .

**Proof.** Every deterministic automaton translates directly into a JMA with a single head. Conversely, a progressing JMA with a single head cannot jump or stay in place, so that it suffices to extend the transition table to make the two final states sink states. (Moreover note that a JMA with a single head can be transformed into a progressing one without adding new heads: it suffices to add a sink state.)  $\blacktriangleleft$

► **Proposition 32** (Proposition 21 in the main text). For all  $k \geq 1$ ,  $JMA(2) \not\subseteq 1DFA(k)$ .

**Proof.** We define a language  $L_{b+}$  that is recognisable by a JMA with only 2 heads while it can be recognised by a  $1DFA$  with  $k$  heads if and only if  $b < \binom{k}{2}$ . To do so, we start from the language defined in [18] to prove the hierarchy theorem for  $1DFAs$ . We slightly modify this language in order to add some information that helps an automaton with jumping heads but not one with only multiple heads.

In [18], it is proven that the language  $L_b = \{w_1\$ \dots \$w_{2b} \mid \forall i, w_i = w_{2b+1-i} \in A^*\}$  is recognisable by a  $1FA$  with  $k$  heads if and only if  $b < \binom{k}{2}$  (with  $\$$  a fresh letter not in  $A$ ). Let us define the language  $L_{b+}$  in the following way:

$$L_{b+} = \{ (+)^b w_1 \$ (+)^{b-1} \$ \dots \$ + w_b \$ w_{b+1} \$ \dots \$ w_{2b} \$ \mid \forall i, w_i \in A^* \wedge w_i = w_{2b+1-i} \}$$

(Again, with  $+$  another fresh letter not in  $A$ .) The proof from [18] can easily be adapted to prove that the language  $L_{b+}$  is recognisable by a  $1DFA$  with  $k$  heads if and only if  $b < \binom{k}{2}$ : it suffices to define type<sup>2</sup> of a configuration according to the number of  $+$  that have been added to the words.

On the other hand we can recognise  $L_{b+}$  with a JMA with only 2 heads. This automaton works in two steps. The first one serves to verify that the word is of the shape  $(+)^b w_1 \$ (+)^{b-1} \$ \dots \$ + w_b \$ w_{b+1} \$ \dots \$ w_{2b} \$$  with  $w_i \in A^*$  for all  $i$ . The second step serves to verify that  $w_i = w_{2b+1-i}$  for all  $i$ . For the first step, one of the heads makes sure the word consists of  $b$  sequences of  $+$  of length  $b, b-1, \dots, 1$ , each followed by a word of  $A^*$  and a  $\$$  separator and then  $b$  words of  $A^*$  followed by a  $\$$  separator. Since  $b$  is a constant of the language this is feasible even if it requires many states. Then the head jumps to the beginning of the word (using the second head). For the second step, one head linearly reads the word. During this process the second head does the following. Each time the first head reads a  $\$$  separator, the second head jumps to the same position if it is also on a  $\$$  separator and rejects otherwise. Then, for each  $+$  read by the first head, the second head goes through the word until it has skipped one  $\$$  separator if it is the first  $+$  of a

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<sup>2</sup> see [18].

sequence and two separators otherwise. Thereby when the first head begins to read the word  $w_i$ , it has just read  $(+)^{b+1-i}$  (or  $(+)^b$  for  $w_1$ ) and thus the second head has skipped  $2(b+1-i) - 1$  separators and is reading the word  $w_{2b+1-i}$ . When the first head begins to read one of the  $w_i$ , the second head begins to read at the same place as the first head until the first head reaches a separator. If at some point the two letters being read differ then the automaton rejects the word. When a separator is reached (simultaneously by the two heads), the second head jumps back to the first one and the process continues. If at some point, after reading a separator, the first head does not read a  $+$  then the process stops and the word is accepted.  $\blacktriangleleft$

► **Lemma 33** (Lemma 23 in the main text). *If  $\mathcal{M}$  is a progressing JMA, the preproof  $\pi_{\mathcal{M}}$  defined in Section 4.1 is valid, and  $[\pi_{\mathcal{M}}] = L(\mathcal{M})$ .*

**Proof.** Assume  $\pi_{\mathcal{M}}$  is not valid, *i.e.* there exists an infinite branch  $\rho$  without validating thread. By considering a sufficiently long prefix of  $\rho$ , we can find addresses  $v, w$  in  $\rho$  such that

- $v$  and  $w$  correspond to the same subtree, and contain a fully labelled sequent.
- there is no thread visiting a  $*$  position from  $v$  to  $w$ .

The path from  $v$  to  $w$  witnesses a loop in the transition graph of  $\mathcal{M}$ . Moreover, since  $*$  positions in  $\pi_{\mathcal{M}}$  are only encountered when a head advances, and a thread is cut only when the corresponding head jumps, this loop does not verify the progressing condition, *i.e.* there is no head that advances without jumping, otherwise it would yield a thread from  $v$  to  $w$  visiting a  $*$  position. We obtain a contradiction, thereby proving that  $\pi_{\mathcal{M}}$  is valid.

To show that  $[\pi_{\mathcal{M}}] = L(\mathcal{M})$ , we can analyse the computation of the proof  $\pi_{\mathcal{M}}$  on a word  $u \in A^*$ . We proceed by induction, and show the computation of  $\pi_{\mathcal{M}}$  closely follows the computation of  $\mathcal{M}$  in the following way:

- each sequence of steps of the evaluation of  $[\pi_{\mathcal{M}}](u)$  between two fully labelled sequents corresponds to a transition of  $\mathcal{M}$
- when reaching address  $v$  with sequent  $(A^*)^{k'}$  fully labelled by  $(s, f, \vec{y})$ , the computation of  $[\pi_{\mathcal{M}}](u)$  evaluates  $[v](u_1, \dots, u_{k'})$  where  $s$  is the current state of  $\mathcal{M}$ ,  $f$  maps each  $u_i$  to a head  $f(i)$ ,  $\vec{y}$  describes the letters currently read by each head, and  $u_1, \dots, u_{k'}$  is the list of suffixes remaining to be read by heads that did not reach the end of the input.

The initial gadget and  $read_1, \dots, read_k$  gadgets ensure that the first fully labelled sequent describes the initial configuration according to the above correspondence. The transition gadget and  $read_i$  gadgets preserve the above invariant, and accurately simulate a transition of  $\mathcal{M}$ . The final gadget allows one to stop the computation of  $[\pi_{\mathcal{M}}](u)$  whenever  $\mathcal{M}$  stops on input  $u$ , and returns the same value.  $\blacktriangleleft$

► **Lemma 34** (Lemma 24 in the main text). *The JMA  $\mathcal{M}$  from Section 4.2 is progressing, and  $L(\mathcal{M}) = [\pi]$ .*

**Proof.** A loop  $(v, r, f) \rightarrow (v, r, f)$  in the transition graph of  $\mathcal{M}$  corresponds to a path in  $\pi$  from  $v$  to some  $vu$ , two addresses corresponding to the same subtree. By validity of  $\pi$ , the branch  $vu^\omega$  contains a validating thread. So there is a thread  $t$  from  $\langle v, i \rangle$  to  $\langle vu, j \rangle$  for some  $i, j$ , containing a  $*$  position. Let  $h = f(i)$  be the index of the head pointing to the  $A^*$  formula in position  $\langle v, i \rangle$ . By construction of  $\delta$ , this head did not jump during the loop (or the thread  $t$  would be cut), and performs at least one action  $\blacktriangleright$  (where the thread  $t$  visits a  $*$  position). We proved that the JMA  $\mathcal{M}$  is progressing, since any loop verifies the progressing condition.

The fact that  $L(\mathcal{M}) = [\pi]$  is proved by induction on the computation of  $[\pi]$  on any input  $u$ . At any point, the construction of  $\mathcal{M}$  preserved the announced invariants: registers store the contents of alphabet formulas, for each  $i$  the content of the  $i^{th}$  star formula is the suffix to the right of the head number  $f(i)$ . The transition function  $\delta$  is built to follow the computation of  $[\pi](u)$  on any  $u$ , and return the same result when a **(tt)** or **(ff)** axiom is reached. ◀